3.5 Implicit Differentiation

So far we have learned to differentiate equations that have been solve ***explicitly*** for y. In this section we will learn how to differentiate functions that do not have ***y explicitly*** in terms of ***x***. Some functions are difficult to solve for $\boldsymbol{y}$ in terms of ***x***, which, in turn, makes it difficult to find the derivative of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$ or $\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$. To overcome this difficulty we will use the method called ***implicit differentiation***.

<u>**Steps to Implicit Differentiation**</u>

1. Differentiate both sides of the equation with respect to ***x***.
2. Solve the equation for ***y'***, or $\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$.

Example: If $x^2 + y^2 = 25$, find y', then find the tangent line to the circle at the point (3, 4).

$\frac{dy}{dx}[x^2 + y^2 = 25]$ We know that $\frac{dy}{dx}x^2 = 2x$ but what about $\frac{dy}{dx}y^2$? Since we are taking the derivative of a $\boldsymbol{y}$ term with respect to $\boldsymbol{x}$ (not y) we follow the derivative with the ***y'*** notation. So … $\frac{dy}{dx}y^2 = 2yy'$ (You can use $\frac{dy}{dx}$ instead of y' if you prefer.) Similarly $\frac{d}{dx}(t^2) = 2tt'$. So now back to our problem

$\frac{dy}{dx}[x^2 + y^2] = \frac{dy}{dx}[25] \;\Rightarrow\; 2x + 2yy' = 0$ Now solve for y'. $y' = -\frac{2x}{2y} \;\Rightarrow\; \boldsymbol{y' = -\frac{x}{y}}$ The slope of the tangent line would be $y'(3, 4) = -\frac{\mathbf{3}}{\mathbf{4}}$. The equation would be $y - y_1 = m(x - x_1)$

$y - 4 = -\frac{3}{4}(\boldsymbol{x} - \mathbf{3}) \;\Rightarrow\; \boldsymbol{y} = -\frac{\mathbf{3}}{\mathbf{4}}\boldsymbol{x} + \frac{\mathbf{25}}{\mathbf{4}}$ is the tangent line to the circle at the point (3, 4).

Example: If $ycos(x) = x^2 + y^2$, find y'. Solutions: $\frac{dy}{dx}[ycos(x)] = \frac{dy}{dx}[x^2 + y^2]$

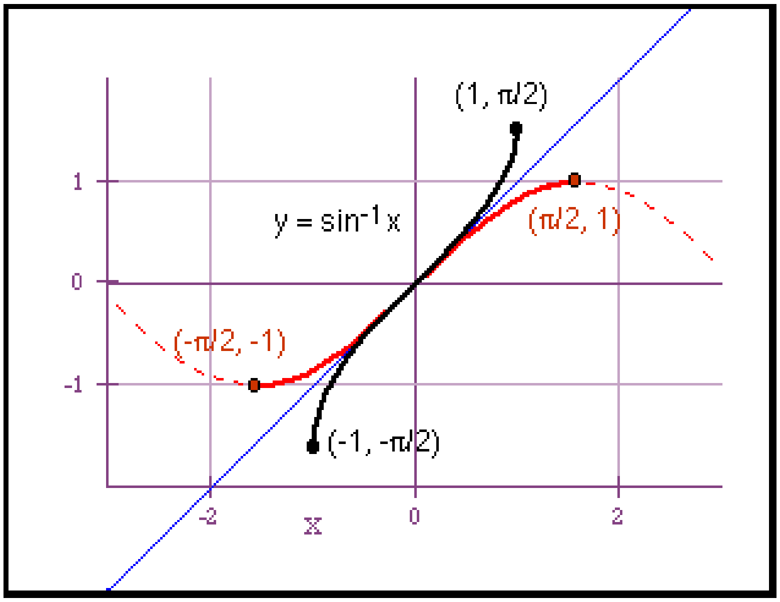
$$y(-\sin(x) + \cos(x)\,y' = 2x + 2yy'$$
$$-ysin(x) + y'\cos(x) = 2x + 2yy' \text{ Solve for } y'.$$
$$y'\cos(x) - 2yy' = 2x + ysin(x)$$
$$y'(\cos(x) - 2y) = 2x + ysin(x)$$
$$\boldsymbol{y' = \frac{2x + ysin(x)}{(\cos(x) - 2y)}}$$

Derivatives of Inverse Trig Functions

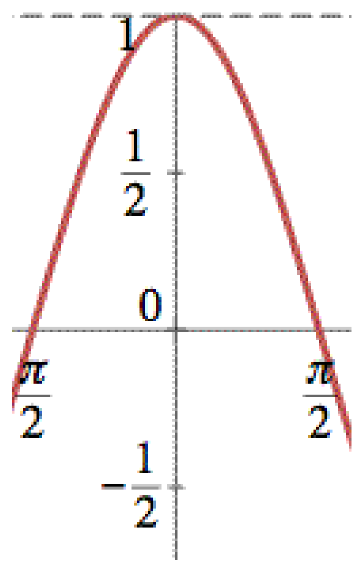
Here we will learn how to find the derivative of the Inverse Trigonometric Functions. If ***f*** is any one-to-one differentiable function, it can be proved that tis inverse function, $\boldsymbol{f^{-1}}$, is also differentiable.

Let's analyze the function **y = sin^{-1}(x).** See the graph below. If **y = sin^{-1}(x)** that also means that **x = sin(y)** has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$y = \sin(x)$ & $y = \sin^{-1}(x)$



$\cos(y)$



Now differentiate **x = sin(y)** implicitly with respect to **x.**

$\frac{dy}{dx}(x) = \frac{dy}{dx}(\sin(y)) \;\Rightarrow\; 1 = \cos(y)\,y'$ (solve for y') $y' = \frac{1}{\cos(y)}$ Now plot cos(y), remember since **$y = \sin^{-1}(x)$** has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, that means that **cos(y)** has a domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Notice that cos(y)>0.

Using the fundamental identity, $\cos^2(y) + \sin^2(y) = 1$

$$\cos^2(y) = 1 - \sin^2(y)$$

$$\cos(y) = \mp\sqrt{1 - sin^2(y)} \text{ but since } \cos(y) > 0$$

$$\cos(y) = \sqrt{1 - sin^2(y)}$$ because **x = sin(y)** we can substitute **x^2** for $\sin^2(y)$

$$\cos(y) = \sqrt{1 - x^2}$$

Now since $y' = \frac{1}{\cos(y)}$ we can write $\boldsymbol{y' = \frac{1}{\sqrt{1-x^2}}}$ Therefore, $\boldsymbol{\frac{dy}{dx}[sin^{-1}] = \frac{1}{\sqrt{1-x^2}}}$

We can do similar methods to show the following:

Derivatives of Inverse Trigonometric Functions:

$\boldsymbol{\frac{d}{dx}[sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}}$ $\boldsymbol{\frac{d}{dx}[cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}}}$ $\boldsymbol{\frac{d}{dx}[tan^{-1}(x)] = \frac{1}{1+x^2}}$

$\boldsymbol{\frac{d}{dx}[csc^{-1}(x)] = -\frac{1}{x\sqrt{x^2-1}}}$ $\boldsymbol{\frac{d}{dx}[sec^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}}$ $\boldsymbol{\frac{d}{dx}[cot^{-1}(x)] = -\frac{1}{1+x^2}}$

Example: Find y' if $y = x\,sec^{-1}(x^3)$. This uses the product rule, chain rule and inverse trig rule.

$$y' = x\left(\frac{1}{x^3\sqrt{x^6-1}}\right)(3x^2) + sec^{-1}(x^3) = \boldsymbol{sec^{-1}(x^3) + \frac{3}{\sqrt{x^6-1}}}$$